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Strong convergence of the modified Ishikawa iterative method for infinitely many nonexpansive mappings in Banach spaces☆

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In this paper, we introduce a new modified Ishikawa iterative process for computing fixed points of an infinite family nonexpansive mapping in the framework of Banach spaces. Then, we establish the strong convergence theorem of the proposed iterative scheme under some mild conditions which solves a variational inequality. The results obtained in this paper extend and improve on the recent results of Qin et al. [Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces, Journal of Computational and Applied Mathematics 230 (1) (2009) 121–127], Cho et al. [Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces, Computers and Mathematics with Applications 56 (2008) 2058–2064] and many others.

1. Introduction

In recent years, the existence of fixed points for finitely or infinitely many nonexpansive mappings has been considered by many authors (see also [1–8]). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [9,10]). The problem of finding an optimal point that minimizes a given cost function over a common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, e.g., [9,11,12]. A simple algorithmic solution to the problem of minimizing a quadratic function over a common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation (see [13,12]). It is an interesting topic for investigating the approximation of fixed points of a family of nonexpansive mappings.

Let E be a real Banach space, C be a closed convex subset of E and $T : C \to C$ be a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. A mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping [14–16]. More precisely, take $t \in (0, 1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1 - t)Tx, \quad \forall x \in C, \tag{1.2}$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [14] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of T. Reich [15] extended

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Browder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$. Xu [16] proved that Reich's results hold in reflexive Banach spaces which have a weakly continuous duality mapping.

Recall that a self-mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|. \tag{1.3}$$

We use Π_C to denote the collection of all contractions on C. That is, $\Pi_C = \{f | f : C \to C \textit{ a contraction}\}$. Note that each $f \in \Pi_C$ has a unique fixed point in C. Throughout the paper we assume that $F(T) \neq \emptyset$. Given a real number $t \in (0, 1)$ and a contraction $f \in \Pi_C$, define another mapping $T_t^f : C \to C$ by

$$T_t^f x = tf(x) + (1 - t)Tx, \quad x \in C.$$

For simplicity we will write T_t for T_t^f provided no confusion occurs.

It is not hard to see that T_t is a contraction on C. Indeed, for $x, y \in C$ we have

$$\begin{aligned} \|T_t x - T_t y\| &= \|t(f(x) - f(y)) + (1 - t)(Tx - Ty)\| \\ &\leq \alpha t\|x - y\| + (1 - t)\|x - y\| \\ &= (1 - t(1 - \alpha))\|x - y\| \\ &\leq \|x - y\|. \end{aligned}$$

Let $x_t := x_t^f \in C$ be the unique fixed point of T_t. Thus x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t.$$

Let A be a strongly positive bounded linear operator on the Hilbert space H [17] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.4}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.5}$$

where S is a nonexpansive mapping and b is a given point in H.

In this paper, we consider the mapping W_n defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \end{cases} \tag{1.6}$$

where $T_1, T_2, \ldots$ is an infinite family of nonexpansive mappings of C into itself and $\lambda_1, \lambda_2, \ldots$ are real numbers such that $0 \leq \lambda_n \leq 1$ for every $n \in \mathbb{N}$.

Recently, Qin et al. [6] proved that the sequences $\{x_n\}$ converge strongly to a common fixed point of the infinite family nonexpansive mappings in Banach spaces under certain appropriate assumptions on the sequences α_n and β_n. Let the sequences $\{x_n\}$ be generated by

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.7}$$

Cho et al. [1] also modified the iterative algorithm (1.7) to have strong convergence by using the viscosity approximation method. They considered the following iterative algorithm:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1 - \beta_n)W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.8}$$

On the other hand, Shang et al. [18] introduced the following new iterative algorithms for a nonexpansive mapping in Hilbert spaces; let the sequences $\{x_n\}$ be generated by

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ z_n = \gamma_n x_n + (1-\gamma_n)Tx_n, \\ y_n = \beta_n x_n + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n \gamma f(x) + (I-\alpha_n A)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.9}$$

They proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T under some mild assumptions.

In this paper, motivated by (1.7)–(1.9), we extend the algorithm (1.9) to an infinite family of nonexpansive mappings in Banach spaces and introduce a composite iterative algorithm as follows:

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ z_n = \gamma_n x_n + (1-\gamma_n)W_n x_n, \\ y_n = \beta_n x_n + (1-\beta_n)W_n z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)y_n, \quad \forall n \geq 0, \end{cases} \tag{1.10}$$

where W_n is defined by (1.6), f is a contraction and A is a strongly positive linear bounded self-adjoint operator. Then, we prove that the sequence $\{x_n\}$ generated by (1.10) converges strongly to a common fixed point.

Next, we consider some special cases of the iterative scheme. If $\{\gamma_n\} = 1$ for all $n \geq 0$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)y_n, \quad \forall n \geq 0. \end{cases} \tag{1.11}$$

If $\{\gamma_n\} = \gamma = 1$ for all $n \geq 0$ and $A = I$ (the identity mapping) in (1.10), then (1.10) reduces to (1.8) of Cho et al. [1]. If $f(x_n) = u$ for all $n \in \mathbb{N}$ in (1.8), then (1.8) reduces to (1.7) of Qin et al. [6]. If $\{\beta_n\} = 0$ and $\{\gamma_n\} = 1$ for all $n \geq 0$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)W_n x_n, \quad \forall n \geq 0. \end{cases} \tag{1.12}$$

If $\{\gamma_n\} = \gamma = 1$ and $\{\beta_n\} = 0$ for all $n \geq 0$ and $A = I$ in (1.10), then (1.10) reduces to

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)W_n x_n, \quad \forall n \geq 0. \end{cases} \tag{1.13}$$

Our results presented in this paper introduce the composite iterative scheme for approximating a fixed point of an infinite family nonexpansive mapping. We also establish the strong convergence of the composite iterative sequences $\{x_n\}$ defined by (1.10), which solves a variational inequality. With an appropriate setting, we obtain the corresponding results due to Qin et al. [6], Cho et al. [1] and many others.

2. Preliminaries

Recall that we let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists a $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex (see also [19]). A Banach space E is said to be *smooth* if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$.

Let E^* be the dual space of E. Let $\varphi : [0, \infty) := \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. This function φ is called a *gauge function*. The duality mapping $J_\varphi : E \to E^*$ associated with a gauge function φ is defined by

$$J_\varphi(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|x\|\varphi(\|x\|), \|f^*\| = \varphi(\|x\|)\}, \quad \forall x \in E,$$

where $\langle ., .\rangle$ denotes the generalized duality pairing. In the case where $\varphi(t) = t$, we write J for J_φ and call J the *normalized duality mapping*.

In a smooth Banach space, we define an operator A as strongly positive [20] if there exists a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \geq \bar{\gamma}\|x\|^2, \qquad \|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J(x)\rangle| \quad a \in [0, 1], b \in [-1, 1],$$

where I is the identity mapping and J is the normalized duality mapping.

If C and D are nonempty subsets of a Banach space E such that C is a nonempty closed convex and $D \subset C$, then a mapping $Q : C \to D$ is sunny [21,22] provided that $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geq 0$ whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions play an important role in our argument. They are characterized as follows [21,22]: if E is a smooth Banach space, then $Q : C \to D$ is a *sunny nonexpansive retraction* if and only if the following inequality holds:

$$\langle x - Qx, J(y - Qx)\rangle \leq 0, \quad \forall x \in C, y \in D. \tag{2.1}$$

Following Browder [23], we say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge φ for which the duality mapping $J_\varphi(x)$ is single-valued and weak-to-weak sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $J_\varphi(x)$ converges weakly to J_φ). It is known that l^p has a weakly continuous duality mapping with a gauge function $\varphi(t) = t^{p-1}$ for all $1 < p < \infty$. Set $\Phi(t) = \int_0^t \varphi(t)\mathrm{d}t, \forall t \geq 0$; then $J_\varphi(x) = \partial\Phi(\|x\|), \forall x \in E$, where ∂ denotes the sub-differential in the sense of convex analysis.

We need the following lemmas for proving our main results.

Lemma 2.1 (*[24]*). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that:

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 (*[7]*). *Let C be a nonempty closed and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 2.1 (*See [4, Remark 3.2]*). It can be found from Lemma 2.2 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$,

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 2.2 (*See [4, Remark 3.3]*). Using Lemma 2.2, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for all $x \in C$. Such a W is called the W-mapping generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$. Since W_n is nonexpansive, then $W : C \to C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.1 that for an arbitrary $\epsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$,

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Lemma 2.3 (*[7]*). *Let C be a nonempty closed and convex subset of a strictly convex Banach space E. Let $T_1, T_2, \ldots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

In 2006, Xu [16] proved that, if E is a reflexive Banach space and has a weakly continuous duality, then there is a sunny nonexpansive retraction from C onto $F(T)$ and it can be constructed as follows.

Lemma 2.4 (*[6, Lemma 1.3.]*). *Let E be a reflexive Banach space that has a weakly continuous duality map $J_\varphi(x)$ with gauge φ. Let C be a closed convex subset of E and let $T : C \longrightarrow C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0, 1)$. Let $x_t \in C$ be the unique solution in C of Eq. (1.2). Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T.*

Under the condition of Lemma 2.4, we define a mapping $Q : C \to F(T)$ by

$$Q(u) := \lim_{t\to 0} x_t \quad \forall u \in C.$$

From Xu ([16], Theorem 3.2), we know that Q is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 2.5 ([25]). *Let E be a uniformly smooth Banach space, C be a closed convex subset of E, $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let us have $f \in \Pi_C$. Then the sequence $\{x_t\}$ defined by*

$$x_t = tf(x_t) + (1-t)Tx_t,$$

converges strongly to a point in $F(T)$. Suppose we define a mapping $Q : \Pi_C \to F(T)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad \forall f \in \Pi_C.$$

Then $Q(f)$ solves the following variational inequality:

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad \forall f \in \Pi_C, p \in F(T). \tag{2.2}$$

In particular, if $f = u \in C$ is a constant, then (2.2) is reduced to the sunny nonexpansive retraction of Reich [15] from C onto $F(T)$,

$$\langle Q(u) - u, J(Q(u)-p)\rangle \leq 0, \quad u \in C, p \in F(T). \tag{2.3}$$

Lemma 2.6 ([26]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.7 ([27,6]). *Assume that a Banach space E has a weakly continuous duality mapping J_φ with gauge φ.*

(i) *For all $x, y \in E$, the following inequality holds:*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle.$$

In particular, for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

(ii) *Assume that a sequence $\{x_n\}$ in E converges weakly to a point $x \in E$.*

Then the following identity holds:

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y-x\|), \quad \forall x, y \in E.$$

Lemma 2.8 ([20]). *Assume that A is a strong positive linear bounded operator on a smooth Banach space E with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

3. Main results

Let E be a Banach space, C a closed convex subset of E, A a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. As previously, let Π_C be the set of all contractions on C. For $t \in (0, 1)$ and $f \in \Pi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)Tx$ on C; that is

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t.$$

In this section, we prove a strong convergence theorem.

Theorem 3.1. *Let C be a nonempty closed and convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, the initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $(0, 1)$, the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;
(iii) $\lim_{n\to\infty} |\gamma_{n+1} - \gamma_n| = 0$.

Then the sequence $\{x_n\}$ generated by (1.10) converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = Q(f)$ and Q is a unique sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$. If we define $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ by

$$Q(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the variational inequality

$$\langle (\gamma f - A)Q(f), J_\varphi(p - Q(f))\rangle \le 0, \quad \forall f \in \Pi_C, p \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.1}$$

In particular, if $f = u \in C$ is a constant, then (3.1) is reduced to the sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$,

$$\langle \gamma u - AQ(u), J_\varphi(p - Q(u))\rangle \le 0, \quad u \in C, p \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.2}$$

Proof. Since $\alpha_n \to 0$ as $n \to \infty$, we may assume, with out loss of generality, that $\alpha_n < (1-\delta_n)\|A\|^{-1}$ for all n. From Lemma 2.8, we know that if $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\bar{\gamma}$. First we show that $\{x_n\}$ is bounded. Let $p \in \bigcap_{n=1}^{\infty} F(T_n)$. By the definition of $\{z_n\}$, $\{y_n\}$ and $\{x_n\}$, we have

$$\begin{aligned} \|z_n - p\| &= \|\gamma_n x_n + (1-\gamma_n)W_n x_n - p\| \\ &\le \gamma_n\|x_n - p\| + (1-\gamma_n)\|W_n x_n - p\| \\ &\le \gamma_n\|x_n - p\| + (1-\gamma_n)\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned}$$

and from this, we have

$$\begin{aligned} \|y_n - p\| &= \|\beta_n x_n + (1-\beta_n)W_n z_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|W_n z_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\le \beta_n\|x_n - p\| + (1-\beta_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n A)y_n - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(y_n - p)\| \\ &\le \alpha_n\|\gamma f(x) - Ap\| + (1 - \alpha_n\bar{\gamma})\|y_n - p\| \\ &\le \alpha_n\|\gamma f(x_n) - \gamma f(p) + \gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &\le \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| + (1 - \alpha_n\bar{\gamma})\|x_n - p\| \\ &= (1 - \alpha_n(\bar{\gamma} - \gamma\alpha))\|x_n - p\| + \alpha_n(\bar{\gamma} - \gamma\alpha)\frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}. \end{aligned}$$

By induction on n, we obtain $\|x_n - p\| \le \max\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\|}{\bar{\gamma} - \gamma\alpha}\}$ for every $n \ge 0$ and $x_0 \in C$; then $\{x_n\}$ is bounded. So, $\{y_n\}$, $\{z_n\}$, $\{W_n x_n\}$, and $\{f(x_n)\}$ are also bounded.

Next, we claim that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Since T_n and $U_{n,n}$ are nonexpansive, from (1.6), we have

$$\begin{aligned} \|W_{n+1}x_n - W_n x_n\| &= \|\lambda_1 T_1 U_{n+1,2}x_n - \lambda_1 T_1 U_{n,2}x_n\| \\ &\le \lambda_1\|U_{n+1,2}x_n - U_{n,2}x_n\| \\ &= \lambda_1\|\lambda_2 T_2 U_{n+1,3}x_n - \lambda_2 T_2 U_{n,3}x_n\| \\ &\le \lambda_1\lambda_2\|U_{n+1,3}x_n - U_{n,3}x_n\| \\ &\;\;\vdots \\ &\le \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \\ &\le M_1\prod_{i=1}^{n}\lambda_i, \end{aligned}$$

where $M_1 \geq 0$ is an appropriate constant such that $\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| \leq M_1$ for all $n \geq 0$. Similarly, we also have $\|W_{n+1}z_n - W_nz_n\| \leq M_2 \prod_{i=1}^{n} \lambda_i$, where $M_2 \geq 0$ such that $\|U_{n+1,n+1}z_n - U_{n,n+1}z_n\| \leq M_2$ for all $n \geq 0$. It follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|(\gamma_{n+1}x_{n+1} + (1-\gamma_{n+1})W_{n+1}x_{n+1}) - (\gamma_n x_n + (1-\gamma_n)W_nx_n)\| \\
&\leq (1-\gamma_{n+1})\|W_{n+1}x_{n+1} - W_{n+1}x_n\| + |\gamma_{n+1} - \gamma_n|\|x_n - W_{n+1}x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&\leq (1-\gamma_{n+1})\|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + \gamma_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&= \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + (1-\gamma_n)\|W_{n+1}x_n - W_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + \|W_{n+1}x_n - W_nx_n\| \\
&\leq \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M_1 \prod_{i=1}^{n} \lambda_i.
\end{aligned}$$

Observe that, on putting $l_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, we have

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 0. \tag{3.3}$$

Now, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1}) + (I - \alpha_{n+1}A)y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1-\beta_{n+1}} + \frac{y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1-\beta_n} - \frac{y_n - \beta_n x_n}{1-\beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}(\gamma f(x_{n+1}) - Ay_{n+1})}{1-\beta_{n+1}} + W_{n+1}z_{n+1} - \frac{\alpha_n(\gamma f(x_n) - Ay_n)}{1-\beta_n} - W_nz_n \right\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|W_{n+1}z_{n+1} - W_nz_n\| \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|W_{n+1}z_{n+1} - W_{n+1}z_n + W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|W_{n+1}z_{n+1} - W_{n+1}z_n\| + \|W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + \|z_{n+1} - z_n\| + \|W_{n+1}z_n - W_nz_n\| \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M_1 \prod_{i=1}^{n} \lambda_i + M_2 \prod_{i=1}^{n} \lambda_i \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&\quad + \|x_{n+1} - x_n\| + |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M \prod_{i=1}^{n} \lambda_i,
\end{aligned}$$

where $M = M_1 + M_2$. Therefore, we have

$$\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| \\
&+ |\gamma_{n+1} - \gamma_n|\|W_{n+1}x_n - x_n\| + M \prod_{i=1}^{n} \lambda_i.
\end{aligned}$$

From the conditions (i), (ii), (iii), and $0 < \lambda_n \leq b < 1$, we obtain

$$\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

It follows from Lemma 2.6, that $\lim_{n\to\infty}\|l_n - x_n\| = 0$. Noting (3.3), we see that

$$\|x_{n+1} - x_n\| = (1-\beta_n)\|l_n - x_n\| \to 0$$

as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.4}$$

Observing that

$$\begin{aligned}\|x_{n+1} - y_n\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n A)y_n - y_n\|\\ &= \alpha_n\|\gamma f(x_n) - Ay_n\|,\end{aligned}$$

and the condition (i), we get

$$\lim_{n\to\infty}\|x_{n+1} - y_n\| = 0. \tag{3.5}$$

On the other hand, we have

$$\|y_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|.$$

Combining (3.4) with (3.5), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.6}$$

Consider

$$\|y_n - W_n z_n\| = \|\beta_n x_n + (1-\beta_n)W_n z_n - W_n z_n\| = \beta_n\|x_n - W_n z_n\|$$

and

$$\begin{aligned}\|z_n - x_n\| &= \|\gamma_n x_n + (1-\gamma_n)W_n x_n - x_n\|\\ &= \|\gamma_n x_n + W_n x_n - \gamma_n W_n x_n - x_n\|\\ &= \|(W_n x_n - x_n) - \gamma_n(W_n x_n - x_n)\|\\ &= \|(1-\gamma_n)(W_n x_n - x_n)\|\\ &= (1-\gamma_n)\|W_n x_n - x_n\|.\end{aligned}$$

It follows that

$$\begin{aligned}\|W_n x_n - x_n\| &\le \|x_n - y_n\| + \|y_n - W_n z_n\| + \|W_n z_n - W_n x_n\|\\ &\le \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\| + \|z_n - x_n\|\\ &= \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\| + (1-\gamma_n)\|W_n x_n - x_n\|.\end{aligned}$$

This implies that

$$\gamma_n\|W_n x_n - x_n\| \le \|x_n - y_n\| + \beta_n\|x_n - W_n z_n\|.$$

From the condition (ii) and (3.6), we get

$$\lim_{n\to\infty}\|W_n x_n - x_n\| = 0.$$

On the other hand, we obtain

$$\|Wx_n - x_n\| \le \|Wx_n - W_n x_n\| + \|W_n x_n - x_n\|.$$

From Remark 2.2 (see also Remark 3.3 of [28]), we have that $\|Wx_n - W_n x_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\|Wx_n - x_n\| = 0. \tag{3.7}$$

Next, we prove that

$$\limsup_{n\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle \le 0. \tag{3.8}$$

By Lemma 2.5, we have the sunny nonexpansive retraction $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle = \limsup_{k\to\infty}\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n_k} - Q(f))\rangle. \tag{3.9}$$

Since E is reflexive, we may assume that $x_{n_k} \rightharpoonup \bar{x}$ for some $\bar{x} \in C$. Since J_φ is weakly continuous, from Lemma 2.7, we have

$$\limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

Put

$$g(x) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - x\|), \quad \forall x \in E.$$

It follows that

$$g(x) = g(\bar{x}) + \Phi(\|x - \bar{x}\|), \quad \forall x \in E.$$

From (3.7), we have

$$\begin{aligned} g(W\bar{x}) &= \limsup_{k\to\infty} \Phi(\|x_{n_k} - W\bar{x}\|) = \limsup_{k\to\infty} \Phi(\|Wx_{n_k} - W\bar{x}\|) \\ &\le \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) = g(\bar{x}). \end{aligned} \tag{3.10}$$

On the other hand, we note that

$$g(W\bar{x}) = \limsup_{k\to\infty} \Phi(\|x_{n_k} - \bar{x}\|) + \Phi(\|W\bar{x} - \bar{x}\|) = g(\bar{x}) + \Phi(\|W\bar{x} - \bar{x}\|). \tag{3.11}$$

Combining (3.10) with (3.11), we obtain

$$\Phi(\|W\bar{x} - \bar{x}\|) \le 0.$$

Hence $W\bar{x} = \bar{x}$ and $\bar{x} \in F(W)$. That is, $\bar{x} \in \bigcap_{n=1}^{\infty} F(T_n)$. Hence, by (3.9) and the sunny nonexpansive retraction from Π_C onto $\bigcap_{n=1}^{\infty} F(T_n)$, we get

$$\limsup_{n\to\infty} \langle \gamma f(x_n) - AQ(f), J_\varphi(x_n - Q(f))\rangle = \langle \gamma f(x_n) - AQ(f), J_\varphi(\bar{x} - Q(f))\rangle \le 0. \tag{3.12}$$

Therefore, we obtain that (3.8) holds.

Finally, we prove that $x_n \to Q(f)$ as $n \to \infty$. Now from Lemma 2.7, we have

$$\begin{aligned} \Phi(\|x_{n+1} - Q(f)\|) &= \Phi(\|\alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n - Q(f)\|) \\ &= \Phi(\|\alpha_n \gamma f(x_n) + (1-\delta_n)y_n - \alpha_n A y_n + \delta_n(y_n - Q(f)) - (1-\delta_n)Q(f) + \alpha_n AQ(f) - \alpha_n AQ(f)\|) \\ &= \Phi(\|\alpha_n \gamma f(x_n) + ((1-\delta_n) - \alpha_n A)y_n + \delta_n(y_n - Q(f)) - ((1-\delta_n) - \alpha_n A)Q(f) - \alpha_n AQ(f)\|) \\ &= \Phi(\|[((1-\delta_n)I - \alpha_n A)(y_n - Q(f)) + \delta_n(y_n - Q(f))] + \alpha_n(\gamma f(x_n) - AQ(f))\|) \\ &\le \Phi(\|((1-\delta_n)I - \alpha_n A)(y_n - Q(f)) + \delta_n(y_n - Q(f))\|) + \alpha_n\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\le \Phi((1-\delta_n - \alpha_n\bar{\gamma})\|y_n - Q(f)\| + \delta_n\|y_n - Q(f)\|) + \alpha_n\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\le \Phi((1-\delta_n - \alpha_n\bar{\gamma})\|y_n - Q(f)\| + \delta_n\|y_n - Q(f)\|) + \alpha_n\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &= (1-\alpha_n\bar{\gamma})\Phi(\|y_n - Q(f)\|) + \alpha_n\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle \\ &\le (1-\alpha_n\bar{\gamma})\Phi(\|x_n - Q(f)\|) + \sigma_n, \end{aligned} \tag{3.13}$$

where $\sigma_n = \alpha_n\langle \gamma f(x_n) - AQ(f), J_\varphi(x_{n+1} - Q(f))\rangle$. By (3.12) and (i), using Lemma 2.1, we see that $\Phi(\|x_n - Q(f)\|) \to 0$ as $n \to \infty$. This implies that $\|x_n - Q(f)\| \to 0$ as $n \to \infty$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a reflexive, smooth and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. Let A be a strongly positive linear bounded self-adjoint operator with coefficient $\bar{\gamma} > 0$ and let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. Assume that $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$, the initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Let $\{x_n\}$ be the composite iterative process defined by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)y_n, \quad \forall n \ge 0. \end{cases} \tag{3.14}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = Q(f)$ and $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $\{\gamma_n\} = 1$ in (1.10), we can reach the desired conclusion easily. □

Corollary 3.3 ([5, Theorem 3.2,]). *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. Let f be a contraction of C into itself with coefficient $\alpha \in (0, 1)$. The initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Let $\{x_n\}$ be the composite iterative process defined by*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n, \quad \forall n \geq 0. \end{cases} \tag{3.15}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = Q(f)$ and $Q : \Pi_C \to \bigcap_{n=1}^{\infty} F(T_n)$ is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $\gamma = 1$ and $A = I$ in (3.14), we can reach the desired conclusion easily. □

Corollary 3.4 ([6, Theorem 2.1,]). *Let C be a nonempty closed convex subset of a reflexive and strictly convex Banach space E which also has a weakly continuous duality map $J_\varphi(x)$ with the gauge φ. Let $T_1, T_2, \ldots$ be a nonexpansive mapping from C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, and let $\lambda_1, \lambda_2, \ldots$ be real numbers. The initial guess $x_0 \in C$ is chosen arbitrarily and the given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are in $(0, 1)$. Let $\{x_n\}$ be the composite iterative process defined by*

$$\begin{cases} x_0 = u \in C \quad \text{chosen arbitrarily}, \\ y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n) y_n, \quad \forall n \geq 0. \end{cases} \tag{3.16}$$

The following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$; *and* $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$;

Then the composite process $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$, where $x^* = Q(u)$ and $Q : C \to \bigcap_{n=1}^{\infty} F(T_n)$ is the unique sunny nonexpansive retraction from C onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. Taking $f(x) = u \in C$ for all $x \in C$ in (3.15), we can reach the desired conclusion easily. □

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